Dimensional Analysis for Planetary Mixer: Modified Power and Reynolds Numbers

G. Delaplace, R. Guérin, and J. C. Leuliet
INRA-LGPTA, 59651 Villeneuve d'Ascq, France

DOI 10.1002/aic.10563
Published online August 29, 2005 in Wiley InterScience (www.interscience.wiley.com).

Mixing times, power consumption, heat transfer, and scale-up predictions in an agitated vessel require the use of correlations between dimensionless groups such as Prandtl, Nusselt, Power, and Reynolds numbers. These dimensionless numbers are now well established for an agitated vessel equipped with a vertically and centrally mounted impeller in the tank for both Newtonian and non-Newtonian fluids. To our knowledge, there is more ambiguity as to the definition of the characteristic speed and dimensions, which should be taken into account in the dimensional analysis of planetary mixers. The aim of this paper is twofold: (1) to propose modified Reynolds and Power numbers for planetary mixers and (2) to ascertain the reliability of the modified dimensionless number proposed for a particular planetary mixer, The TRI-AXE®, which uses a combination of rotation and gyration of a pitched blade turbine to achieve mixing. The modified Reynolds and Power numbers proposed involve the maximum tip speed as characteristic velocity and are consistent with the definition of traditional Reynolds and Power numbers when only a single revolution around the vertical axis of the mixing device occurs in the vessel, as is the case for a standard mixing system. Experimental power measurements carried out with a planetary mixer when mixing highly viscous Newtonian fluids show that the modified Reynolds and Power numbers proposed succeed in obtaining a unique power curve for the mixing system independently of the speed ratio. This close agreement proves that the modified Reynolds and Power numbers are well adapted for engineering purposes and can be used to compare the power-consumption performances of planetary mixers with well-established technologies. © 2005 American Institute of Chemical Engineers *AIChE J,* 51: 3094–3100, 2005

Keywords: mixing, planetary mixers, dimensional analysis, power consumption, Newtonian viscous fluids

Introduction

In the second half of the twentieth century, the systematic use of dimensional analysis to investigate mixing processes has allowed this field to evolve from "arts into sciences." Today the whole field of classical stirring technology (here, the word *classical* refers to impellers vertically and centrally mounted in the tank) has been examined, so that the definition of significant dimensionless groups has now become well established. Thus, for any stirring operation (heat transfer, blending, and so on), depending on the flow regime and the mixing systems investigated, numerous correlations involving various dimensionless numbers have been proposed in the open literature for both design and/or scale-up. Consequently, a deeper process understanding and/or better or reproducible products can be achieved. This is not yet the case for the planetary mixers. Indeed, for this kind of mixing equipment, the literature is relatively scarce.[1-6] In addition, some dimensionless numbers still suffer from ambiguity with respect to the characteristic speed and dimension, which should be taken into account.

This lack prevents the comparison of performances of the planetary mixers with those of classical mixing systems. For

Correspondence concerning this article should be addressed to G. Delaplace at delapla@lille.inra.fr.

example, the Reynolds number *Re* and Power number N_p are now well established for an agitated vessel equipped with various geometries of impeller vertically and centrally mounted in the tank for both Newtonian and non-Newtonian fluids. N_p and *Re* are traditionally used to characterize the power demands of a classical mixing system (N_p vs. *Re*), when no vortex formation occurs in the tank (the Froude number can be neglected). In these dimensionless numbers, the characteristic length chosen is the diameter of the impeller that, in fact, corresponds to an external dimension of the impeller perpendicular to the vertical revolution axis. The characteristic velocity chosen ($=ND$) is proportional to the maximum linear velocity encountered in the vessel ($=\pi ND$). Note that the characteristic velocity corresponds to the impeller tip speed divided by π.

However, for planetary mixers the maximum linear velocity encountered in the vessel is influenced by both the relative and displacement velocities and, consequently, depends on the two revolution speeds. In the same way, the characteristic dimension of the mixing system is much more complex to define. Therefore the classical Reynolds and Power numbers should be adequately modified to take into account the complexity of the combined motion required by the agitator.

The aim of this paper is twofold : (1) to propose modified Reynolds and Power numbers for planetary mixers and (2) to ascertain the reliability of the dimensionless numbers proposed for a new planetary mixer, The TRIAXE® system, which uses a combination of rotation and gyration of a pitched blade turbine to achieve mixing.

Experimental

Mixing equipment

The mixing equipment used in this investigation is the TRIAXE® system (HOGNON® S.A., Mormant, France), which allows the agitator to combine two motions: gyration and rotation (Figure 1). This planetary mixer is characterized by two revolutionary motions that are nearly perpendicular. Gyration is a revolution of the agitator around a vertical axis, whereas rotation is a revolution of the agitator around a nearly horizontal axis. These double motions allow the agitator to periodically come in contact with the entire volume of the vessel. In this work, the mixing tool for the TRIAXE® system is a four pitched blade turbine (Figure 2). The tank used is a transparent glass cylinder with rounded bottom (Figure 2). The diameter of the vessel is 0.4 m. In this work, experimental measurements were carried out when the agitator was fully immersed in the liquid. This liquid height corresponds to a liquid volume of 38 L, corresponding to a liquid height of 0.39 m.

The mixing equipment was driven by two variable-speed motors. To obtain the total power consumption of the TRIAXE® system, power draw measurements were carried out alternately for the two variable-speed drive motors that control the impeller revolutions. To do so, a torque meter (Scaime Inc., Annemasse Cedex, France) ranging from 0 to 5 N·m, was mounted alternately on the two motor drive shafts (Figure 2) and the torque was measured for various impeller speed ratios. Total power draw was estimated as a simple summation of the two motor drives. More exactly, the procedure to obtain power requirement of each motor was the following:

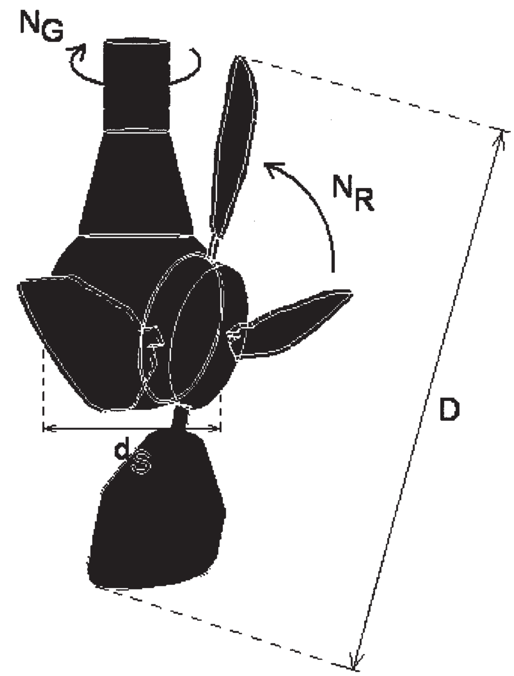


Figure 1. TRIAXE® system.

For the planetary mixer studied, $d_s = 0.14$ m and $D = 0.38$ m.

(1) The torquemeter is mounted on one of the two motor drive shafts just before the reduction gearbox. The reduction gearbox ratio R_G for the gyration motor drive is equal to 145. The reduction gearbox ratio R_R for the rotation motor drive is equal to 34.

(2) The revolution speed of the motor drive shaft that is not equipped with the torque meter is maintained constant, whereas torque measurements exerted on the second motor drive shaft ($M_{G\,motor}$ or $M_{R\,motor}$, respectively) are performed for various revolution speeds of the second motor drive shaft ($N_{G\,motor}$ or $N_{R\,motor}$, respectively). This was done both when mixing a Newtonian fluid ($M_{G\,motor}$ or $M_{R\,motor}$, respectively) or when air (no loading) was contained in the vessel ($M_{G0\,motor}$ or $M_{R0\,motor}$, respectively), to determine the effective torque exerted on each motor drive shaft [($M_{G\,motor} - M_{G0\,motor}$) or ($M_{R\,motor} - M_{R0\,motor}$), respectively].

(3) Torque measurements were carried out on the full range of speeds available on the motor drive shafts. Because of the gearbox reductions on the two motor drive shafts, the range of revolution speeds available for the two agitator shafts (after the gearbox) are: 0 to 16 rpm for N_G and 0 to 90 rpm for N_R.

(4) Then neglecting mechanical friction in the gear box, the effective torque required by each agitator shaft was plotted: $(M_G - M_{G0}) = (M_{G\,motor} - M_{G0\,motor})R_G$ or $(M_R - M_{R0}) = (M_{R\,motor} - M_{R0\,motor})R_R$, respectively, as a function of the agitator speeds (respectively N_G or N_R). At this step, we have observed that effective torque values exerted on rotation agitator shaft were not dependent on the gyrational speed of the impeller (see Figure 3 as an example) and vice versa. Consequently, the effective torque values exerted on rotation agitator shaft (respectively on gyration agitator shaft) can be predicted by the only knowledge of viscosity and rotational impeller speed: $M_R - M_{R0} = 1.265\mu N_R$ (respectively $M_G - M_{G0} = 1.595\mu N_G$). Note that, for a fixed revolution speed and the

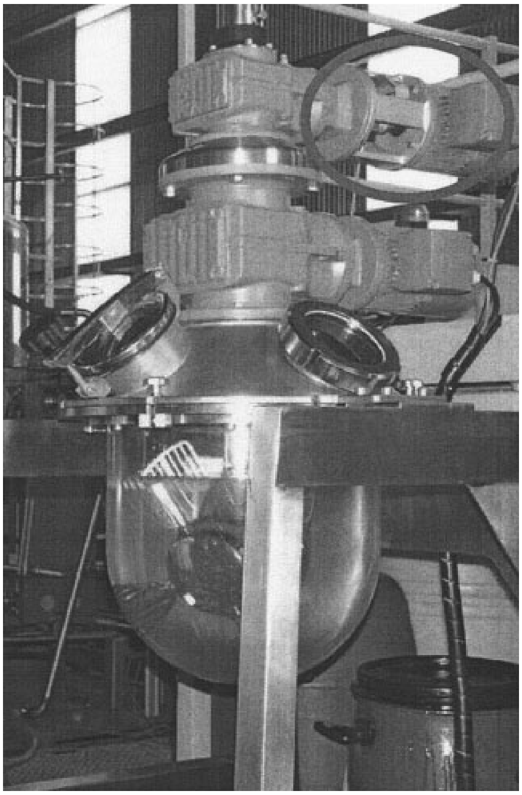


Figure 2. Picture of the agitator and vessel of the TRIAXE® system used in this investigation.

Red circle on the picture refers to the torquemeter device. The vessel tank diameter T and liquid height H_L were set equal to 0.4 and 0.39 m, respectively. For the planetary mixer studied, $d_s = 0.14$ m and $D = 0.38$ m.

TRIAXE® system under investigation, the torque exerted on gyration agitator shaft is the same order as that obtained for rotation agitator shaft. The same observation was already made and detailed in a previous paper[7] when the agitator was not fully immersed in the liquid. Then, the power input for each motor can be deduced. Power attributed to gyration and rota-

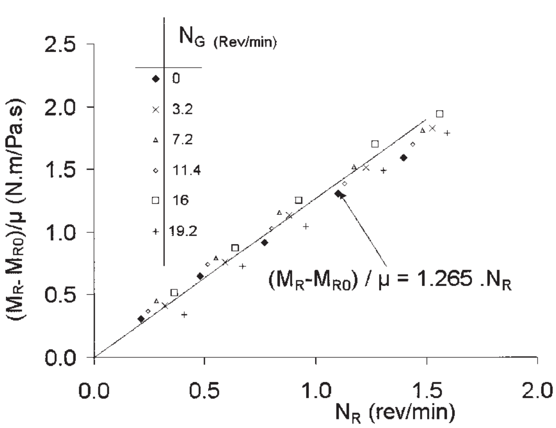


Figure 3. Effective torque exerted on rotation/μ vs. impeller rotational speed.

Symbols refer to various impeller gyrational speeds tested. Solid line is obtained by linear regression.

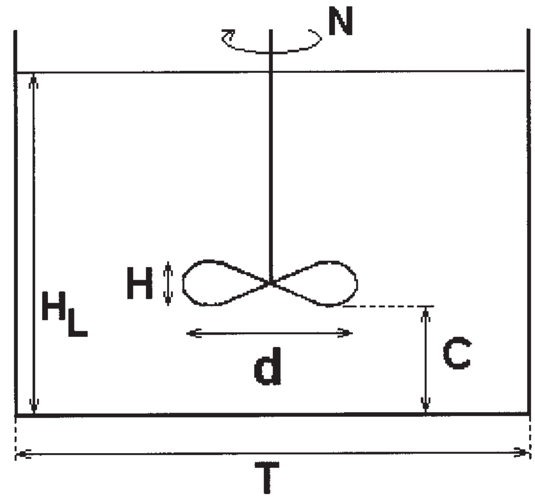


Figure 4. Classical geometric parameters and notations used for mixing vessel equipped with an impeller vertically and centrally mounted in the tank.

tion are, respectively, $P_G = (M_{G\,motor} - M_{G0\,motor})2\pi N_{G\,motor}$ and $P_R = (M_{R\,motor} - M_{R0\,motor})2\pi N_{R\,motor}$, where $N_{G\,motor}$ and $N_{R\,motor}$ are, respectively, the gyrational and rotational speeds of the motor shaft. Finally, total power consumption P can be estimated by adding the contribution of power draw required by the two motors ($P_G + P_R$).

(5) Using this procedure, the total power consumption of the planetary mixer when mixing a Newtonian fluid ($\mu = 22$ Pa·s and $\rho = 1400$ kg/m^3) was plotted in Figures 8 and 9 (see below).

Agitated fluid

The agitated fluid is a highly viscous Newtonian fluid. The liquid consists of glucose syrup/water mixtures of various viscosities ranging from 15 to 29 Pa·s, depending on the temperature encountered in the vessel. The rheological properties and densities of the test fluid were obtained at the same temperatures as those encountered in the mixing equipment. Density at 20°C of the test fluid is 1400 kg/m^3. The rheological properties of the test fluid were measured in standard controlled rotational speed concentric cylinders (Rheomat 30, Contraves AG, Zurich, Switzerland). The shear rate range applied to the controlled shear rate viscosimeter varied from 0 to 500 s^{-1}and corresponds to the shear rate range encountered in the tank.

Theory

Dimensional analysis for an agitated vessel equipped with an impeller vertically and centrally mounted in the tank

When mixing Newtonian liquids in an agitated vessel without baffles, equipped with an impeller vertically and centrally mounted in the tank (Figure 4), the power P of a given stirrer type and given installation conditions (vessel diameter T, agitator height H, liquid height H_L, and bottom clearance C) in a homogeneous liquid depends on the agitator diameter d (as the characteristic length), the material parameters of the liquid

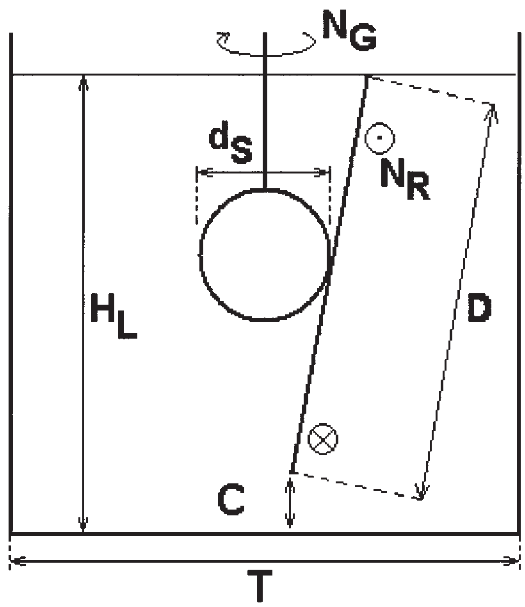


Figure 5. Sketch and symbols used for the TRIAXE® system.

(density ρ and dynamic viscosity μ), and on the stirrer speed N. When no vortex exists, the acceleration attributed to gravity g is not relevant with respect to the power draw.

The above-mentioned dependency on dimensional parameters

$$F_1(P, d, T, H, H_L, C, \rho, \mu, N) = 0 \tag{1}$$

leads, by way of dimensional analysis, to the following dependency between dimensionless numbers[8]:

$$F_2(N_p, \mathrm{Re}, T/d, H/d, H_{L/d}, C/d) = 0 \tag{2}$$

$N_p = P/\rho N^3 d^5$ is the so-called Newton number (often termed the *Power number*) and $\mathrm{Re} = \rho N d^2/\mu$ is the Reynolds number.

For a given agitator and fixed installation conditions, Eq. 2 is reduced to a dependency between N_p and Re.

$$F_3(N_p, \mathrm{Re}) = 0 \tag{3}$$

Dimensional analysis of a planetary mixer

Assuming that the acceleration arising from gravity (as in the previous case) does not influence the mixing process of highly viscous fluids, the list of relevant dimensional parameters influencing the power consumption, when mixing Newtonian liquids with the TRIAXE® system equipped with a pitched blade turbine, is

$$F_4(P, D, d_s, \rho, \mu, N_R, N_G, \textit{installation conditions}) = 0 \tag{4}$$

In Eq. 4, N_R and N_G are, respectively, the rotational and the gyrational impeller speeds; D and d_s are the geometric parameters reported in Figure 5. ρ and μ are, respectively, the density and dynamic viscosity of the liquid. The installation conditions refer here to the vessel diameter T, the liquid height H_L, and the bottom clearance C.

For fixed installation conditions, this seven-parameter dimensional space leads to a power characteristic consisting of four pi-numbers

$$F_5\left(N_{pG}, \mathrm{Re}_G, \frac{N_R}{N_G}, \frac{D}{d_s}\right) = 0 \tag{5}$$

where $N_{p_G} = P/\rho N_G^3 d_s^5$ and $\mathrm{Re}_G = \rho N_G d_s^2/\mu$.

For a given planetary system, the ratio D/d_s is fixed and the power characteristic is reduced to a mutual dependency between each of the following three parameters constituting the pi-set

$$F_6\left(N_{pG}, \mathrm{Re}_G, \frac{N_R}{N_G}\right) = 0 \tag{6}$$

Note that in Eqs. 5 and 6 the characteristic length and velocity that appear in the modified power N_{p_G} and Reynolds Re_G numbers are d_s and $N_G d_s$, respectively. This was chosen at first approximation, given that when no rotational motion occurs ($N_R = 0$), the planetary mixer under investigatiom is transformed to a classical mixing system (Figure 6). Using symbols adopted for classical mixing systems, d_s, D, and N_G (Figure 6) become, respectively, equal to d, H, and N (Figure 4). Thus, modified power N_{p_G} and Reynolds Re_G numbers transform to the well-known N_p and Re traditionally used for classical mixing systems.

A closer look at Eq. 4 facilitates a reduction in the number of physical quantities in the list of relevant parameters. Indeed, it is possible to introduce as a characteristic velocity u_{ch} a value that is proportional to the impeller tip speed and reduce the list of relevant physical variables describing the problem by three parameters: N_R, N_G, and D:

$$F_7(P, d_s, \rho, \mu, u_{ch}, \textit{installation conditions}) = 0 \tag{7}$$

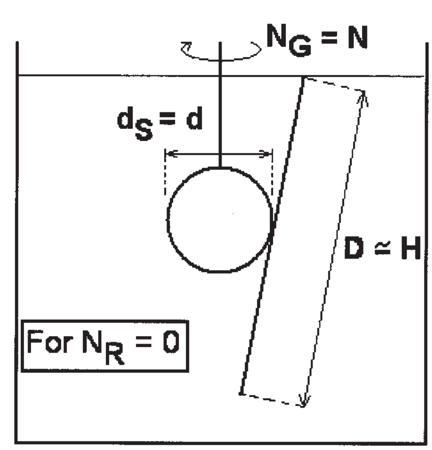


Figure 6. Analogies between the geometric parameters of the classical mixing system and the TRIAXE® planetary mixer when rotational speed is set at zero.

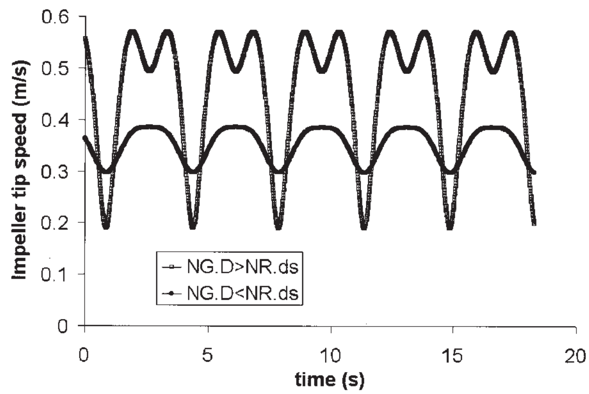


Figure 7. Example of evolution with time of the absolute impeller tip speed.

For the two symbols, D =0.38 m and d_s = 0.14 m. For the black symbols, N_R and N_G were set at 0.286 and 0.1 rev/s, respectively. For the gray symbols, N_R and N_G were set at 0.286 and 0.345 rev/s, respectively.

For fixed installation conditions and a given planetary system (D/d_s is known), this five-parameter dimensional space leads to a power characteristic consisting of only two pi-numbers

$$F_8(N_{p_m}, \mathrm{Re}_m) = 0 \tag{8}$$

with $N_{p_m} = P/\rho u_{ch}^3 d_s^2$ and $\mathrm{Re}_m = \rho u_{ch} d_s/\mu$.

Note that for a planetary mixer, the instantaneous impeller tip speed $u_{\text{impeller tip}}(t)$ in an inert reference frame, is not a constant value with time t. This will be shown and discussed later. By analogy with a dimensional analysis for classical mixing systems, the characteristic velocity chosen in Eq. 8 was the maximum velocity encountered in the vessel divided by π

$$u_{ch} = \max[u_{\text{impeller tip}}(t)]/\pi \tag{9}$$

To sum up, the dimensional analysis of the power consumption of the TRIAXE® system in a fixed installation condition when mixing highly viscous fluids, leads either to a relationship between three pi-numbers (Eq. 6) or two pi-numbers (Eq. 8) if a characteristic speed proportional to the maximum impeller tip speed is introduced in the parametric dimensional space. In addition, it has been shown that the pi-numbers can be reduced to well-known Reynolds and Power numbers, when the rotational speed is zero. This is quite logical because, in this case, the TRIAXE® system is transformed to a classical mixing system equipped with an agitator vertically and centrally mounted in the tank.

In the following section, the way to compute the characteristic speed for the TRIAXE® system will be detailed. Then, the reliability of the two pi-numbers proposed will be ascertained, using power consumption measurements.

Determination of the characteristic speed of the TRIAXE® system

For the TRIAXE® system, which combines the dual motions reported in Figure 1, the magnitude of instantaneous impeller tip speed in an inert reference frame is defined as follows (more details are given in Appendix A)

$$u_{\text{impeller tip}}(t) = \left\{\left(2\pi N_R \frac{D}{2}\right)^2 + \left(2\pi N_G \frac{d_s}{2}\right)^2 + \left(2\pi N_G \frac{D}{2}\right)^2 \cos^2(2\pi N_R t) - 2\pi^2 N_R N_G d_s D \sin(2\pi N_R t)\right\}^{1/2} \tag{10}$$

where N_R and N_G are, respectively, the rotational and the gyrational impeller speeds; D and d_s are the geometric parameters reported in Figure 1; and t represents the time.

Depending on whether the ratio $N_R d_s/N_G d$ is <1 (see Appendix B), the function $t \mapsto u_{\text{impeller tip}}(t)$ does not have the same evolution with time and reaches two ($N_R d_s/N_G d > 1$) or four extrema ($N_R d_s/N_G d < 1$). The various shapes of the absolute impeller tip speed with time are illustrated in Figure 7. The derivation of Eq. 10 allows us to obtain the instants for which the magnitude of absolute impeller tip speed reaches extrema (see Appendix B), and thus to compute the maximum value for impeller tip speed $\max[u_{\text{impeller tip}}(t)]$. Finally, using Eq. 9, u_{ch} can be deduced.

Results and Discussion

Power consumption measurements obtained for the TRIAXE® system investigated when mixing a Newtonian fluid are shown in Figure 8. Results are presented in terms of the dimensionless numbers (N_{p_G} vs. Re_G) previously defined in Eq. 6.

Figure 8 clearly shows that the power characteristic of the TRIAXE® system cannot be reduced to a unique relation between Power and Reynolds numbers when d_s and $N_G d_s$ are used as characteristic length and velocity, respectively. Indeed in this case, it would appear that the power data are strongly influenced by the speed ratio values N_R/N_G.

In contrast, it is shown in Figure 9 that the use of modified Reynolds and Power numbers, such as those suggested in Eq. 8, succeed in obtaining a unique power characteristic for the TRIAXE® system, independently of the speed ratios chosen. This proves to a certain extent the reliability of using maximum impeller tip speed as characteristic velocity and a dimension perpendicular to the vertical axis of revolution as characteristic length. Moreover, it can be observed from Figure 9 that the product $N_{p_m}\mathrm{Re}_m$ is constant, as obtained with classical mixing systems when mixing highly viscous fluids. The value of the product was found to be 343. As with classical mixing equip-

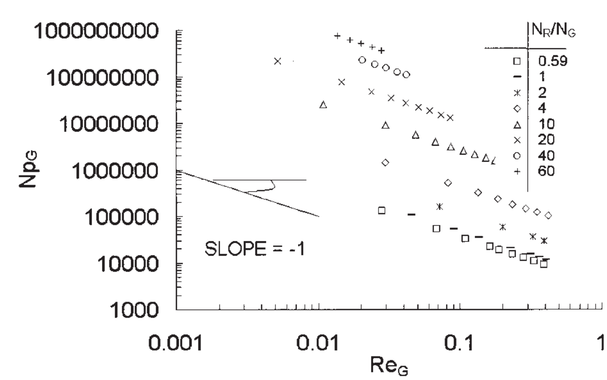


Figure 8. Power characteristics of the planetary mixer using the three pi-numbers defined in Eq. 6.

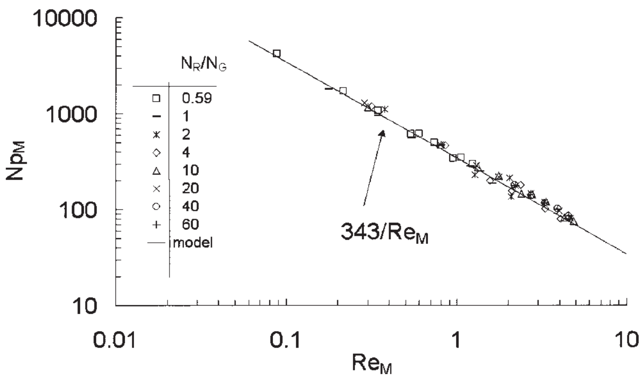


Figure 9. Power characteristic of the planetary mixer using the two pi-numbers defined in Eq. 8.

ments, these results allow a Reynolds number to be defined and the power consumption of the planetary mixer to be predicted, on the basis of the two impeller revolution speeds (gyration and rotation) and the liquid properties. It should be noted that a product $N_{p_m}\mathrm{Re}_m$ of 343, does not mean that the power consumption of the planetary mixer is fairly similar to that of the helical ribbon impeller (N_pRe around 300 for standard helical ribbon[9]). Indeed, for a given liquid under a laminar regime, power consumption is obtained not only by multiplying the power constant N_pRe, but also by the characteristic length exponent 3. In this case, to include the same tank volume, the characteristic length required for the helical ribbon is around twofold higher than that of the planetary mixer under investigation. Consequently, the power draw required by the TRIAXE® system is lower than that by standard helical ribbon impeller.

Conclusion

In this paper, modified Reynolds and Power numbers for a planetary mixer—the TRIAXE® system, which combines dual revolutionary motions—has been developed.

It has been shown that the proposed modified Reynolds and Power numbers, which involve the maximum impeller tip speed as characteristic velocity and a dimension perpendicular to the vertical axis of revolution as characteristic length, allow one to obtain a unique power characteristic of the mixing system, regardless of the speed ratios. Moreover, the characteristic length and velocity chosen ensure that the modified dimensionless numbers proposed are consistent with the definition of traditional Reynolds and Power numbers, when the impeller performed only a single revolution around the vertical axis in the tank, as in the case of a classical mixing system. So, it clearly appeared that the modified Reynolds and Power numbers can be easily used to compare the power consumption of planetary mixers with that of conventional mixing systems.

In our judgment, the suggested modified Power and Reynolds numbers can be used for other planetary mixers that combine two revolutions around a vertical axis. Only the expression of the characteristic velocity must be modified, to allow for the differing operating conditions. In contrast to double-armed planetary mixers or twin-blade arrangement, dimensional analysis defined in Eq. 6 should be preferred because there is no way to reduce the pi-numbers using a characteristic velocity that takes into account the two different revolutionary speeds.

Acknowledgments

The authors are grateful to F. Brisard and J. F. Dauphin for valuable work in obtaining the experimental data and figures presented here.

Literature Cited

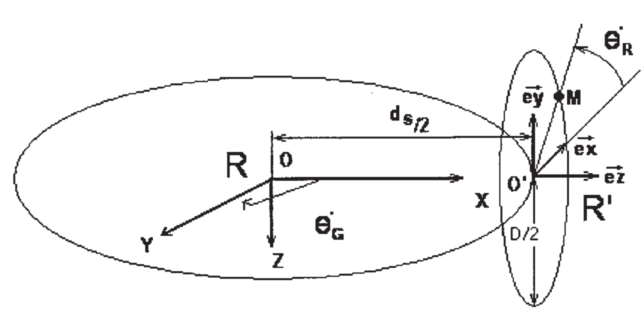
1. Tanguy PA, Bertrand F, Labrie R, Brito de la Fuente E. Numerical modelling of the mixing of viscoplastic slurries in a twin blade planetary mixer. *Trans IChemE* 1996;74:499-504.
2. Tanguy PA, Thibault F, Dubois C, Aït-Kadi A. Mixing hydrodynamics in a double planetary mixer. *Trans IChemE.* 199;77:318-323.
3. Landin M, York P, Cliff MJ, Rowe RC. Scaleup of a pharmaceutical granulation in planetary mixers. *Pharm Dev Technol.* 1999;4:145-150.
4. Zhou G, Tanguy PA, Dubois C. Power consumption in a double planetary mixer with non-Newtonian and viscoelastic material. *Trans IChemE.* 2000;78:445-453.
5. Jongen T. Characterization of batch mixers using numerical flow simulations. *AIChE J.* 2000;46:2140-2150.
6. Delaplace G, Bouvier L, Moreau A, Guérin R, Leuliet J-C. Determination of mixing time by colourimetric diagnosis—Application to a new mixing system. *Exp Fluids.* 2004;36:437-443.
7. Delaplace G, Guérin R, Bouvier L, Moreau A, Leuliet J-C. Performances of a novel mixing device: The TRIAXE® system—Mixing times and power consumption for highly viscous fluids. *Proceedings of the 11th European Conference on Mixing, Bamberg, Germany*. Düsseldorf, Germany: VDI–GVC; 2003:589-596.
8. Zlokarnik M. *Stirring Theory and Practice*. Weinheim, Germany: Wiley–VCH Verlag; 2001.
9. Delaplace G, Leuliet J-C, Relandeau V. Circulation and mixing times for helical ribbon impeller: Review and experiments. *Exp Fluids.* 2000;28:170-182.

Appendix A Scheme 1

Considering M to be a point located at the TRIAXE® impeller tip, the absolute velocity of M in the rotating reference frame ($\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$) is given by

$$\vec{V}a = \left(\frac{d\overrightarrow{OM}}{dt}\right)_R = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_R + \left(\frac{d\overrightarrow{O'M}}{dt}\right)_R$$

$$\left(\frac{d\overrightarrow{OO'}}{dt}\right)_R = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_{R'} + \vec{\Omega} \wedge \overrightarrow{OO'} \qquad \text{with} \qquad \vec{\Omega} = \vec{\Omega}_{R'/R}$$



Scheme 1.

$$\left(\frac{d\overrightarrow{OO'}}{dt}\right)_{R'} = \vec{0} \qquad \text{and} \qquad \vec{\Omega} \wedge \overrightarrow{OO'}$$

$$= \begin{vmatrix} 0 \\ -2\pi N_G \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 0 \\ 0 \\ d_s/2 \end{vmatrix} = \begin{vmatrix} -2\pi N_G d_s/2 \\ 0 \\ 0 \end{vmatrix}$$

$$\left(\frac{d\overrightarrow{O'M}}{dt}\right)_R = \left(\frac{d\overrightarrow{O'M}}{dt}\right)_{R'} + \vec{\Omega} \wedge \overrightarrow{O'M} \quad \text{with} \quad \vec{\Omega} = \vec{\Omega}_{R'/R}$$

$$\overrightarrow{O'M} = \begin{vmatrix} (D/2)\cos(2\pi N_R t) \\ (D/2)\sin(2\pi N_R t) \\ 0 \end{vmatrix}$$

$$\left(\frac{d\overrightarrow{O'M}}{dt}\right)_{R'} = \begin{vmatrix} -(D/2)2\pi N_R \sin(2\pi N_R t) \\ (D/2)2\pi N_R \cos(2\pi N_R t) \\ 0 \end{vmatrix}$$

$$\vec{\Omega} \wedge \overrightarrow{O'M}$$

$$= \begin{vmatrix} 0 \\ -2\pi N_G \\ 0 \end{vmatrix} \wedge \begin{vmatrix} (D/2)\cos(2\pi N_R t) \\ (D/2)\sin(2\pi N_R t) \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 2\pi N_G (D/2)\cos(2\pi N_R t) \end{vmatrix}$$

$$\vec{V}a = \begin{vmatrix} -(D/2)2\pi N_R \sin(2\pi N_R t) - 2\pi N_G d_s/2 \\ (D/2)2\pi N_R \cos(2\pi N_R t) \\ 2\pi N_G (D/2)\cos(2\pi N_R t) \end{vmatrix}$$

Consequently, the absolute velocity of M in the inert reference frame ($\vec{X}$, $\vec{Y}$, $\vec{Z}$) is given by

$$\vec{V}a = \begin{vmatrix} [-(D/2)2\pi N_R \sin(2\pi N_R t) - 2\pi N_G d_s/2]\sin(2\pi N_G t) + [2\pi N_G (D/2)\cos(2\pi N_R t)]\cos(2\pi N_G t) \\ -[-(D/2)2\pi N_R \sin(2\pi N_R t) - 2\pi N_G d_s/2]\cos(2\pi N_G t) + [2\pi N_G (D/2)\cos(2\pi N_R t)]\sin(2\pi N_G t) \\ -(D/2)2\pi N_R \cos(2\pi N_R t) \end{vmatrix}$$

Thus, the magnitude of absolute velocity of M is given by

$$\|\vec{V}a\| = u_{\text{impeller tip}}(t) = \sqrt{\begin{array}{l}[(D/2)2\pi N_R)^2 + (2\pi N_G(d_s/2)]^2 - 2(D/2)2\pi N_R 2\pi N_G (d_s/2)\sin(2\pi N_R t) \\ + [2\pi N_G (D/2)]^2\cos^2(2\pi N_R t)\end{array}}$$

Appendix B

It can be shown that the function $t \mapsto u_{\text{impeller tip}}(t)$, defined in Eq. 10, reaches an extremum when the derivative of the function $t \mapsto w(t)$ is equal to zero with $w(t)$ given by $w(t) = -2(D/2)2\pi N_R 2\pi N_G(d_s/2)\sin(2\pi N_R t) + [2\pi N_G(D/2)]^2\cos^2(2\pi N_R t)$.

The derivative of the function $t \mapsto w(t)$ is $dw(t)/dt = 2\pi N_R\cos(2\pi N_R t)\{[-2(D/2)2\pi N_R 2\pi N_G(d_s/2)] - 2[2\pi N_G(D/2)]^2\sin(2\pi N_R t)\}$.

Consequently $dw(t)/dt = 0$ when the following conditions apply:

$$t = \frac{1}{4N_R}$$

$$t = \frac{3}{4N_R}$$

$$\sin(2\pi N_R t) = -\frac{N_R d_s}{N_G d}$$

So, depending on whether ratio $N_R d_s/N_G d$ is <1, function $t \mapsto u_{\text{impeller tip}}(t)$ has two or four extrema that are respectively reached at

$$t = \left\{\frac{1}{4N_R}, \frac{3}{4N_R}\right\}$$

or at

$$t = \left\{\frac{1}{4N_R}, \frac{3}{4N_R}\right\}$$

or

$$\arcsin\left(-\frac{N_R d_s}{N_G d}\right), \qquad \left[\pi - \arcsin\left(-\frac{N_R d_s}{N_G d}\right)\right]$$

From these extrema, a maximum value for impeller tip speed $\max[u_{\text{impeller tip}}(t)]$ can be selected.

Manuscript received July 7, 2004, and revision received Apr. 4, 2005.